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Fluctuations and Phase Transition Dynamics [1]

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Abstract

Kibble and Zurek have provided a unifying causal picture for the appearance of classical defects like cosmic strings or vortices at the onset of phase transitions in relativistic QFT and condensed matter systems respectively. In condensed matter the predictions are partially supported by agreement with experiments in superfluid helium. We provide an alternative picture for the initial appearance of defects that supports the experimental evidence. When the original predictions fail, this is understood, in part, as a consequence of thermal fluctuations (noise), which play a comparable role in both condensed matter and QFT.

[1] To be published in the proceedings of the 4th Conference on Quantum and stochastic gravity, String cosmology and Inflation, Peyresq, September 1999

1 Overview

In this talk I want to consider the emergence of 'classical' field configurations - topological defects - after a phase transition, and the extent to which thermal fluctuations can inhibit this process. This is of particular interest in the early universe, for which we expect a sequence of transitions from a very symmetric initial state, and in which the presence of classical defects can have important astrophysical consequences.

The relevance of topological defects is that when symmetry breaks, it does not do so uniformly. At the very least, the field is uncorrelated on the scale of the causal horizon at any time. Since a broken symmetry is, necessarily, characterised by degenerate vacua, the choice of different vacua in domains in which the fields are uncorrelated will lead naturally to topological defects between them as the field does its best to order on large scales. The nature of the defects depends on the relevant homotopy group of the ground-state manifold. The most acceptable defect on cosmological grounds is the 'cosmic string' - a generalised field vortex - which may have played in role in structure formation.

Given a theory that permits vortices, at some time after the transition we expect to find a network of them, behaving essentially classically as Nambu-Goto strings, intersecting, chopping off loops which decay, and straightening segments to reduce field gradients. Simple calculations suggest late-time scaling solutions, with similarity on a wide range of scales.

The details do not concern us here. What interests us is how this collection of essentially classical objects, which can be observed directly, in principle, came into existence. The simplest question, that we shall address here, is what is the density of cosmic strings (or other defects) at the time of their appearance?

The early universe is very hot, but such a problem requires us to go beyond equilibrium Thermal Field Theory. In practice, we often know remarkably little about the dynamics of thermal systems. For simplicity, I shall assume scalar field order parameters, with *continuous* transitions. In principle, the field correlation length diverges at a continuous transition. In practice, it does not since there is not enough time. One possibility is that the separation of 'defects' is characterised by the correlation length when it checks its growth. If this were simply so, a measurement of defect densities would be a measurement of correlation lengths. Estimates of this early field ordering and its contingent defects in the early universe have been made by Kibble[1, 2], using simple thermal[1] arguments or causal arguments[2] different from the one above (although that is also due to Kibble[3]).

There are great difficulties in converting such predictions for the early universe into experimental observations since, but for a possible stray monopole, we have no direct evidence for them having existed [2]. Zurek suggested[4] that similar arguments to those in [2] were applicable to condensed matter systems for which direct experiments on defect densities could be performed. This has lead to considerable activity from theorists working on the boundary between QFT and Condensed Matter theory and from condensed

[2] Although this does not impede our ability to make predictions for defect-driven fluctuations in the CMB, for example

matter experimentalists. To date almost all experiments have involved superfluids, for which vortices can be produced readily. All but one experiment is in agreement with these simple causal predictions and we shall pay particular attention to this one failure of prediction. In this talk I shall

- review the Kibble/Zurek causality predictions for initial correlation lengths and defect densities.
- summarise the results of the condensed matter experiments and present an alternative picture for the onset of defect production for condensed matter systems. I shall then show how this alternative picture gives essentially the same results as the Zurek picture for those condensed matter systems for which there is experimental agreement.
- provide an explanation for why some condensed matter experiments will be in disagreement with Zurek's predictions, including the experiment in question. We shall suggest that the prediction fails, in part, because of the presence of thermal noise.
- use these ideas to address the more complicated problem of the appearance of 'classical' defect configurations in QFT in the light of Kibble's predictions, and the role of thermal noise in them.

2 When symmetry breaks, how big are the smallest identifiable pieces?

Defects in the large-scale ordering of the field can only appear once the transition has taken place. If it is the case that defect density can be identified simply from the field correlation length, the *maximum* density (an experimental observable in condensed matter systems, although not for the early universe) will be associated with the *smallest* identifiable correlation length in the broken phase once the transition has been effected. This provides the initial condition for the evolution of field ordering. From this viewpoint, we can observe the defects by default merely by determining the correlation length at that time.

In order to see how to identify these 'smallest pieces'[3] it is sufficient to consider the simplest theory with vortices, that of a single relativistic *complex* scalar field in three spatial dimensions, undergoing a temperature quench. In the first instance we assume that the qualitative dynamics of the transition are conditioned by the field's *equilibrium* free energy, of the form

$$F(T) = \int d^3x \, \left(|\nabla\phi|^2 + m^2(T)|\phi|^2 + \lambda|\phi|^4 \right) \tag{2.1}$$

[3]The title of this section is essentially that posed in recent papers by Zurek[5].

Prior to the transition, at temperature $T > T_c$, the critical temperature, $m(T) > 0$ plays the role of an effective 'plasma' mass due to the interactions of ϕ with the heat bath, which includes its own particles. After the transition, when T is effectively zero, $m^2(0) = -M^2 < 0$ enforces the $U(1)$ symmetry-breaking, with field expectation values $\langle|\phi|\rangle = \eta$, $\eta^2 = M^2/\lambda$. The change in temperature that leads to the change in the sign of m^2 is most simply understood as a consequence of the system expanding. Thus, in the early universe, a weakly interacting relativistic plasma at temperature $T \gg M$ has an entropy density $s \propto T^3$. As long as thermal equilibrium can be maintained, constant entropy S per comoving volume, $S \propto sa(t)^3$, gives $T \propto a(t)^{-1}$ and falling, for increasing scale factor $a(t)$. Models that attempt to take inflation into account, however, lead to 'preheating' that is not Boltzmannian[6]. Nonetheless, even in such cases it is possible to isolate an effective temperature for long-wavelength modes. This is all that is necessary, but is too sophisticated for the simple scenarios that we shall present here. We shall not even include a metric in Eq.2.1.

The minima of the final potential of Eq.2.1 now constitute the circle $\phi = \eta e^{i\alpha}$. When the transition starts ϕ begins to fall into the valley of the potential, choosing a random phase. This randomly chosen phase can vary from point to point subject to continuity. At late times the failure of the field to be uniform in phase on large scales will lead to it twisting around classical 'defects' - solutions to $\delta F/\delta\phi = 0$ that locally minimise the energy stored in field gradients and potentials. Those of interest to us are vortices, tubes of 'false' vacuum $\phi \approx 0$, around which the field phase changes by $\pm 2\pi$. In an early universe context these are the simplest possible 'cosmic strings'.

How this collapse takes place determines the size of the first identifiable domains. It was suggested by Kibble and Zurek that this size is essentially the equilibrium field correlation length ξ_{eq} at some appropriate temperature close to the transition. I shall argue later that this is too simple but, nonetheless, it is a plausible starting point. Two very different mechanisms have been proposed for estimating this size.

2.1 Thermal activation

In the early work on the cosmic string scenario an alternative possibility to simple causality was to assume[1] that initial domain size was fixed in the Ginzburg regime by the correlation length at that time, rather than the causal radius. By this we mean the following. Once we are below T_c, and the central hump in $V(\phi) = m^2(T)|\phi|^2 + \lambda|\phi|^4$ is forming, T_G signals the temperature above which there is a significant probability for thermal fluctuations over the central hump on the scale of the correlation length. Most simply, it is determined by the condition

$$\Delta V(T_G)\xi_{eq}^3(T_G) \approx T_G \tag{2.2}$$

where $\Delta V(T)$ is the difference between the central maximum and the minima of $V(\phi, T)$. We find $|1 - T_G/T_c| = O(\lambda)$.

Whereas, above T_G there will be a population of 'domains', fluctuating in and out of existence, at temperatures below T_G fluctuations from one minimum to the other become increasingly unlikely. When this happens the correlation length is

$$\xi_{eq}(T_G) = O\Big(\frac{\xi_0}{\sqrt{1 - T_G/T_c}}\Big), \tag{2.3}$$

where $\xi_0 = M^{-1}$ is the natural unit of length, the Compton wavelength of the ϕ particles.

It is tempting[1, 7] to identify $\xi_{eq}(T_G)$ with the scale at which stable domains begin to form. We shall see later that this is incorrect, for quenches that are not too slow. However, some care is needed if (as can happen in condensed matter physics) we never leave the Ginzburg regime.

The formation of large domains is an issue that requires more than equilibrium physics. The most simple dynamical arguments can be understood in terms of causality.

2.2 Causality

We have already mentioned that causality puts an *upper* bound on domain size. Specifically, if $G(r,t)$ is the two-field correlation function at time t for separation r, then G vanishes for $r \geq 2t$ approximately. This was used by Kibble[3] to put an upper bound on monopole density in the early universe. If this causal bound and the Ginzburg criteria attempt to set scales once the critical temperature has been *passed*, the causal arguments considered now attempt to set scales *before* it is reached.

Here we attempt a *lower* bound on domain size, an upper bound on defect density. Suppose the temperature $T(t)$ varies sufficiently slowly with time t that it makes sense to replace $V(\phi, T)$ by $V(\phi, T(t))$. With $m^2(T(t))$ vanishing at $T = T_c$, which we suppose happens at $t = 0$, the *equilibrium* correlation length of the field fluctuations $\xi_{eq}(T(t)) = |m^{-1}(T(t))|$ diverges at $T(t) = T_c$. It is sufficient to adopt a mean-field approximation in which $m^2(T) \propto (T - T_c)$. The true correlation length $\xi(t)$ cannot diverge like $\xi_{eq}(T(t))$, since it can only grow so far in a finite time.

Initially, for $t < 0$, when we are far from the transition, we again assume effective equilibrium, and the field correlation length $\xi(t)$ tracks $\xi_{eq}(T(t))$ approximately. However, as we get closer to the transition $\xi_{eq}(T(t))$ begins to increase arbitrarily fast. As a crude upper bound, the true correlation length fails to keep up with $\xi_{eq}(T(t))$ by the time $-\bar{t}$ at which ξ_{eq} is growing at the speed of light, $d\xi_{eq}(T(-\bar{t}))/dt = 1$. It was suggested by Kibble[2] that, once we have reached this time $\xi(t)$ *freezes* in, remaining approximately constant until the time $t \approx +\bar{t}$ after the transition when is once again becomes comparable to the now *decreasing* value of ξ_{eq}. The correlation length $\xi_{eq}(\bar{t}) = \xi_{eq}(-\bar{t})$ is argued to provide the scale for the minimum domain size *after* the transition.

Specifically, if we assume a time-dependence $m^2(t) = -M^2 t/t_Q$ in the vicinity of $t = 0$, when the transition begins to be effected, then the causality condition gives $t_C = t_Q^{1/3}(2M)^{-2/3}$. As a result,

$$M\xi_{eq}(\bar{t}) = (M\tau_0)^{1/3}, \tag{2.4}$$

which, with condensed matter in mind, we write as

$$\bar{\xi} = \xi_{eq}(\bar{t}) = \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{1/3} \tag{2.5}$$

where $\tau_0 = \xi_0 = M^{-1}$ are the natural time and distance scales. In contrast to Eq.2.3, Eq.2.5 depends explicitly on the quench rate, as we would expect.

2.3 QFT or Condensed Matter

This approach of Kibble was one of the motivations for a similar analysis by Zurek[4] of transitions in condensed matter. Qualitatively, neither the Ginzburg thermal fluctuations, with fluctuation length Eq.2.3, nor the simple causal argument above depend critically on the fact that the free energy Eq.2.1 is originally assumed to be derived from a relativistic *quantum* field theory. After rescaling, F could equally well be the Ginzburg-Landau free energy for the complex order-parameter field whose magnitude determines the superfluid density. That is,

$$F(T) = \int d^3x \left(\frac{\hbar^2}{2m}|\nabla\phi|^2 + \alpha(T)|\phi|^2 + \frac{1}{4}\beta|\phi|^4\right), \tag{2.6}$$

in which $\alpha(T) \propto m^2(T)$ vanishes at the critical temperature T_c. The only difference is that, in the causal argument, the speed of light should be replaced by the speed of (second) sound, with different critical index.

Explicitly, let us assume the mean-field result $\alpha(T) = \alpha_0\epsilon(T)$, where $\epsilon = (T/T_c - 1)$, remains valid as T/T_c varies with time t. In particular, we first take $\alpha(t) = \alpha(T(t)) = -\alpha_0 t/\tau_Q$ in the vicinity of T_c. The fundamental length scale ξ_0 is given from Eq.2.6 as $\xi_0^2 = \hbar^2/2m\alpha_0$. The Gross-Pitaevski theory suggests a natural time-scale $\tau_0 = \hbar/\alpha_0$. When, later, we adopt the time-dependent Landau-Ginzburg (TDLG) theory we find this still to be true, empirically, at order-of-magnitude level, and we keep it.

It follows that the equilibrium correlation length $\xi_{eq}(t)$ and the relaxation time $\tau(t)$ diverge when t vanishes as

$$\xi_{eq}(t) = \xi_0 \left|\frac{t}{\tau_Q}\right|^{-1/2}, \ \tau(t) = \tau_0 \left|\frac{t}{\tau_Q}\right|^{-1}. \tag{2.7}$$

The speed of sound is $c(t) = \xi_{eq}(t)/\tau(t)$, slowing down as we approach the transition as $|t|^{1/2}$. The causal counterpart to $d\xi_{eq}(t)/dt = 1$ for the relativistic field is $d\xi_{eq}(t)/dt = c(t)$. This is satisfied at $t = -\bar{t}$, where $\bar{t} = \sqrt{\tau_Q\tau_0}$, with corresponding correlation length

$$\bar{\xi} = \xi_{eq}(\bar{t}) = \xi_{eq}(-\bar{t}) = \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{1/4}. \tag{2.8}$$

(cf. Eq.2.5). A variant of this argument that gives essentially the same results is obtained by comparing the quench rate directly to the relaxation rate of the field fluctuations. We stress that, yet again, the assumption is that the length scale that determines the initial correlation length of the field freezes in *before* the transition begins.

3 Experiments

The end result of the simple causality arguments is that, both for QFT and condensed matter, when the field begins to order itself its correlation length has the form

$$\bar{\xi} = \xi_{eq}(\bar{t}) = \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{\gamma}. \tag{3.9}$$

for appropriate γ. [4]

The jump that Kibble made[2] in QFT was to assume that the correlation length Eq.3.9 also sets the scale for the typical minimum intervortex distance. That is, the *initial* vortex density n_{def} is[5] is assumed to be

$$n_{def} = \frac{1}{f^2}\frac{1}{\bar{\xi}^2} = \frac{1}{f^2\xi_0^2}\left(\frac{\tau_0}{\tau_Q}\right)^{2\gamma}. \tag{3.10}$$

for $\gamma = 1/3$ and $f = O(1)$. We stress that this assumption is *independent* of the argument that lead to Eq.2.5. Since ξ_0 also measures cold vortex thickness, $\tau_Q \gg \tau_0$ corresponds to a measurably large number of widely separated vortices.

Even if cosmic strings were produced in so simple a way in the very early universe it is not possible to compare the density Eq.3.10 with experiment, in large part because of our uncertainty as to what is the appropriate theory. It was Zurek who first suggested that this causal argument for defect density be tested in condensed matter systems.

3.1 Superfluid helium

Vortex lines in both superfluid ^4He and ^3He are good analogues of global cosmic strings. In ^4He the bose superfluid is characterised by a complex field ϕ, whose squared modulus $|\phi|^2$ is the superfluid density. The Landau-Ginzburg theory for ^4He has, as its free energy, $F(T)$ of Eq.2.6. The static classical field equation $\delta F/\delta\phi = 0$ has vortex solutions as before. Specifically, a simple (winding number unity) static straight vortex along the z-axis has the form

$$\Phi(\mathbf{x}) = h(r)e^{\pm i\theta}, \tag{3.11}$$

where $\theta = \arctan(y/x)$ and $r^2 = x^2 + y^2$. For small r, $h(r) = O(r)$, and for large r, $h(r) = \eta(1 - O(\xi_0^2/r^2))$, with effective width ξ_0.

The situation is more complicated, but more interesting, for ^3He. The reason is that ^3He is a *fermion.* Thus the mechanism for superfluidity is very different from that of ^4He. Somewhat as in a BCS superconductor, these fermions form the counterpart to Cooper pairs. However, whereas the (electron) Cooper pairs in a superconductor form a ^1S state, the ^3He pairs form a ^3P state. The order parameter $A_{\alpha i}$ is a complex 3×3 matrix $A_{\alpha i}$.

[4] In fact, the powers of Eq.2.5 and Eq.2.8 are mean-field results, changed on implementing the renormalisation group.

[5] Equivalently, the length of vortices in a box volume v is $O(n_{def}v)$.

There are two distinct superfluid phases, depending on how the $SO(3) \times SO(3) \times U(1)$ symmetry is broken. If the normal fluid is cooled at low pressures, it makes a transition to the $^{3}He - B$ phase.

The Landau-Ginzburg free energy is, necessarily, more complicated, permitting many types of vortex[8], but the effective potential $V(A_{\alpha i}, T)$ has the diagonal form[9] $V(A, T) = \alpha(T)|A_{ai}|^2 + O(A^4)$ for small fluctuations, and this is all that we need for the production of vortices at very early times. Thus the Zurek analysis leads to the prediction Eq.3.10, as before, for appropriate γ.

3.2 Counting vortices

Although ^{3}He is more complicated to work with, the experiments to check Eq.3.10 are cleaner, since even individual vortices can be detected by magnetic resonance. Second, because vortex width is many atomic spacings the Landau-Ginzburg theory is good ($\gamma = 1/4$).

So far, experiments have been of two types. In the Helsinki experiment[10] superfluid ^{3}He in a rotating cryostat is bombarded by slow neutrons. Each neutron entering the chamber releases 760 keV, via the reaction $n +^{3} He \rightarrow p +^{3} He + 760 keV$. The energy goes into the kinetic energy of the proton and triton, and is dissipated by ionisation, heating a region of the sample above its transition temperature. The heated region then cools back through the transition temperature, creating vortices. Vortices above a critical size grow and migrate to the centre of the apparatus, where they are counted by an NMR absorption measurement. The quench is very fast, with $\tau_Q/\tau_0 = O(10^3)$. Agreement with Eq.3.10 and Eq.2.8 is good, at the level of an order of magnitude. This is even though it is now argued[11] that the Helsinki experiment should *not* show agreement because of the geometry of the heating event.

The second type of experiment has been performed at Grenoble and Lancaster[12]. Rather than count individual vortices, the experiment detects the total energy going into vortex formation. As before, ^{3}He is irradiated by neutrons. After each absorption the energy released in the form of quasiparticles is measured, and found to be less than the total 760 keV. This missing energy is assumed to have been expended on vortex production. Again, agreement with Zurek's prediction Eq.3.10 and Eq.2.8 is good.

The experiments in ^{4}He, conducted at Lancaster, follow Zurek's original suggestion. The idea is to expand a sample of normal fluid helium so that it becomes superfluid at essentially constant temperature. That is, we change $1 - T/T_c$ from negative to positive by reducing the pressure and increasing T_c. As the system goes into the superfluid phase a tangle of vortices is formed, because of the random distribution of field phases. The vortices are detected by scattering second sound off them, and its attenuation gives a good measure of vortex density. A mechanical quench is slow, with τ_Q some tens of milliseconds, and $\tau_Q/\tau_0 = O(10^{10})$[6]. Two experiments have been performed[13, 14]. In

[6] For ^{4}He mean-field theory is poor, and a better value for γ is $\gamma = 1/3$.

the first fair agreement was found with the prediction Eq.3.10, but the second experiment has failed to see any vortices whatsoever.

There is certainly no agreement, in this or any other experiment on ^3He, with the thermal fluctuation density that would be based on Eq.2.3.

4 The Kibble-Zurek picture for the value of $\bar{\xi}$ is correct.

To do better than the simple causality arguments we need a concrete model for the dynamics.

4.1 Condensed matter: the TDLG equation at early times

We assume that the dynamics of the transition can be derived from the explicitly time-dependent Landau-Ginzburg free energy

$$F(t) = \int d^3x\,\left(\frac{\hbar^2}{2m}|\nabla\phi|^2 + \alpha(t)|\phi|^2 + \frac{1}{4}\beta|\phi|^4\right). \tag{4.12}$$

obtained from Eq.2.6 on identifying $\alpha(t) = \alpha(T(t)) = \alpha_0\epsilon(t)$, where $\epsilon = (T/T_c - 1)$. In a quench in which T_c or T changes it is convenient to shift the origin in time, to write $\epsilon(t)$ as

$$\epsilon(t) = \epsilon_0 - \frac{t}{\tau_Q}\theta(t) \tag{4.13}$$

for $-\infty < t < \tau_Q(1+\epsilon_0)$, after which $\epsilon(t) = -1$. ϵ_0 measures the original relative temperature and τ_Q defines the quench rate. The quench begins at time $t = 0$ and the transition from the normal to the superfluid phase begins at time $t = \epsilon_0\tau_Q$. Times subsequent to that are defined by $\Delta t = t - t_0$.

Motivated by Zurek's later numerical[5] simulations, we adopt the time-dependent Landau-Ginzburg (TDLG) equation for F, on expressing ϕ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, that

$$\frac{1}{\Gamma}\frac{\partial\phi_a}{\partial t} = -\frac{\delta F}{\delta\phi_a} + \eta_a, \tag{4.14}$$

where η_a is Gaussian thermal noise, satisfying

$$\langle\eta_a(\mathbf{x},t)\eta_b(\mathbf{y}',t')\rangle = 2\delta_{ab}T(t)\Gamma\delta(\mathbf{x}-\mathbf{y})\delta(t-t'). \tag{4.15}$$

This is a crude approximation for ^4He, and a simplified form of a realistic description of ^3He but it is not a useful description of QFT, as it stands.

It is relatively simple to determine the validity of Zurek's argument since it assumes that freezing in of field fluctuations occurs just before symmetry breaking begins. At that time the effective potential $V(\phi, T)$ is still roughly quadratic and we can see later that,

for the relevant time-interval $-\bar{t} \leq \Delta t \leq \bar{t}$ the self-interaction term can be neglected ($\beta = 0$).

In space, time and temperature units in which $\xi_0 = \tau_0 = k_B = 1$, Eq.4.14 then becomes

$$\dot{\phi}_a(\mathbf{x}, t) = -[-\nabla^2 + \epsilon(t)]\phi_a(\mathbf{x}, t) + \bar{\eta}_a(\mathbf{x}, t). \tag{4.16}$$

where $\bar{\eta}$ is the renormalised noise. The solution of the 'free'-field linear equation is straightforward, giving a Gaussian equal-time correlation function[15, 16]

$$\langle \phi_a(\mathbf{r}, t)\phi_b(\mathbf{0}, t)\rangle = \delta_{ab} G(\mathbf{r}, t) \tag{4.17}$$

where

$$G(r, t) = \int_0^\infty d\tau \, \bar{T}(t - \tau/2) \left(\frac{1}{4\pi\tau}\right)^{3/2} e^{-r^2/4\tau} \, e^{-\int_0^\tau ds \, \epsilon(t-s/2)}. \tag{4.18}$$

and $\bar{T}$ is the renormalised temperature. At time $t_0 = \epsilon_0 \tau_0$, when the transition begins, a saddle-point calculation shows that, provided the quench is not too fast,

$$G(r, t_0) \approx \frac{T_c}{4\pi r} \, e^{-a(r/\bar{\xi})^{4/3}}, \tag{4.19}$$

where $a = O(1)$, confirming Zurek's result.

Zurek's prediction is robust, since further calculation shows that $\xi(t)$ does not vary strongly in the interval $-\bar{t} \leq \Delta t \leq \bar{t}$, where $\Delta t = t - t_0$.

4.2 QFT: Closed time-path ensemble averaging at early times

For QFT the situation is rather different. In the previous section, instead of working with the TDLG equation, we could have worked with the equivalent Fokker-Planck equation for the probability $p_t^{FP}[\Phi]$ that, at time $t > 0$, the measurement of ϕ will give the function $\Phi(\mathbf{x})$. Thus $G(r, t)$ of Eq.4.17 can be written as

$$\delta_{ab} G(\mathbf{r}, t) = \langle \phi_a(\mathbf{r}, t)\phi_b(\mathbf{0}, t)\rangle = \int \mathcal{D}\Phi \, p_t^{FP}[\Phi]\Phi_a(\mathbf{r})\Phi_b(\mathbf{0}). \tag{4.20}$$

When solving the dynamical equations for a hot quantum field it is convenient to work with probabilities from the start. Taking $t = 0$ as our starting time for the evolution of the complex field ϕ suppose that, at this time, the system is in a pure state, in which the measurement of ϕ would give $\Phi_0(\mathbf{x})$. That is:-

$$\hat{\phi}(t = 0, \mathbf{x})|\Phi_0, t = 0\rangle = \Phi_0|\Phi_0, t = 0\rangle. \tag{4.21}$$

The probability $p_t[\Phi]$ that, at time $t > 0$, the measurement of ϕ will give the function $\Phi(\mathbf{x})$, is the double path integral

$$p_t[\Phi] = \int_{\phi_\pm(0)=\Phi_0}^{\phi_\pm(t)=\Phi} \mathcal{D}\phi_+ \mathcal{D}\phi_- \; \exp\Big\{ i\Big(S_t[\phi_+] - S_t[\phi_-] \Big) \Big\}, \tag{4.22}$$

where $\mathcal{D}\phi_\pm = \mathcal{D}\phi_{\pm,1}\mathcal{D}\phi_{\pm,2}$) and $S_t[\phi]$ is the (time-dependent) action obtained from Eq.2.1, on substituting $m(t) = m(T(t))$ for $m(T)$.

$p_t[\Phi]$ can be written in the closed time-path form in which, instead of separately integrating $\phi_\pm$ along the time paths $0 \leq t \leq t_f$, the integral can be interpreted as time-ordering of a field ϕ along the closed path $C_+ \oplus C_-$ of Fig.1, where $\phi = \phi_+$ on C_+ and $\phi = \phi_-$ on C_-. When we extend the contour from t_f to $t = \infty$ either ϕ_+ or ϕ_- is an equally good candidate for the physical field, but we choose ϕ_+.

The choice of a pure state at time $t = 0$ is too simple to be of any use. For simplicity, we assume that Φ is Boltzmann distributed at time $t = 0$ at an effective temperature of $T_0 = \beta_0^{-1}$ according to the Hamiltonian $H[\Phi]$ corresponding to the free-field action $S[\phi]$, obtained by setting $\lambda = 0$ in Eq.2.1, in which ϕ is taken to be periodic in imaginary time with period β_0. We now have the explicit form for $p_t[\Phi]$,

$$p_t[\Phi] = \int_B \mathcal{D}\phi \, e^{iS_C[\phi]} \, \delta[\phi_+(t_f) - \Phi], \tag{4.23}$$

written as the time ordering of a single field along the contour $C = C_+ \oplus C_- \oplus C_3$, extended to include a third imaginary leg, where ϕ takes the values ϕ_+, ϕ_- and ϕ_3 on C_+, C_- and C_3 respectively, for which S_C is $S[\phi_+]$, $S[\phi_-]$ and $S_0[\phi_3]$.

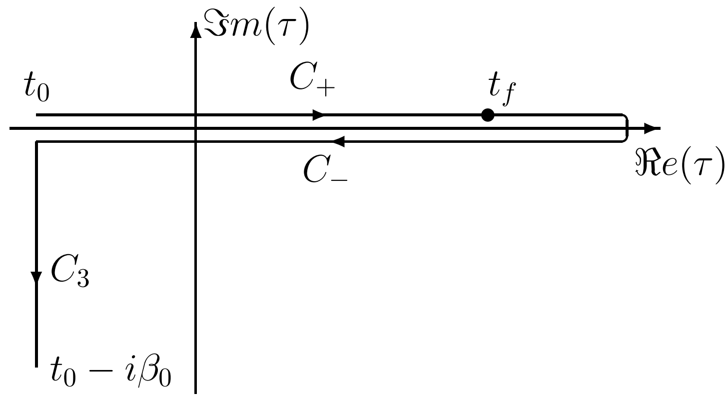

Figure 1: The closed timepath contour $C_+ \oplus C_-$, with the Boltzmann imaginary leg

Just as we had no need to calculate $p^{FP}[\Phi]_t$ explicitly in condensed matter we can average in QFT without having to calculate $p_t[\Phi]$ explicitly. Specifically,

$$G_{ab}(r,t) = \langle \Phi_a(\mathbf{r})\Phi_b(\mathbf{0})\rangle_t = \int \mathcal{D}\Phi \, p_t[\Phi]\Phi_a(\mathbf{r})\Phi_b(\mathbf{0}) \tag{4.24}$$

is given by

$$G_{ab}(r,t) = \langle \phi_a(\mathbf{r},t)\phi_b(\mathbf{0},t)\rangle, \tag{4.25}$$

the equal-time thermal Wightman function with the given thermal boundary conditions.

Fortunately, as for the condensed matter case, the interval $-\bar{t} \leq \Delta t \leq \bar{t}$ occurs in the *linear* regime, when the self-interactions are unimportant. The relevant equation for constructing the correlation functions of this one-field system is now the second-order equation

$$\frac{\partial^2 \phi_a}{\partial t^2} = -\frac{\delta F}{\delta \phi_a}, \tag{4.26}$$

for F of Eq.2.1. This is solvable in terms of the mode functions $\chi_k^{\pm}(t)$, identical for $a = 1, 2$, satisfying

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t)\right]\chi_k^{\pm}(t) = 0, \tag{4.27}$$

subject to $\chi_k^{\pm}(t) = e^{\pm i\omega_{in} t}$ at $t \leq 0$, for incident frequency $\omega_{in} = \sqrt{\mathbf{k}^2 + \epsilon_0 M^2}$ and $m^2(t) = \epsilon(t) M^2$, where $\epsilon(t)$ is parameterised as for the TDLG equation above. This corresponds to a temperature quench from an initial state of thermal equilibrium at temperature $T_0 > T_c$, where $(T_0/T_c - 1) = \epsilon_0$. The diagonal correlation function $G(r,t)$ of Eq.4.17 is given as the equal-time propagator

$$\begin{aligned} G(r,t) &= \int d\!\!\!/^3k\, e^{i\mathbf{k}.\mathbf{x}} \chi_k^{+}(t)\chi_k^{-}(t) C(k) \\ &= \frac{1}{2\pi^2}\int dk\, k^2 \frac{\sin kr}{kr} \chi_k^{+}(t)\chi_k^{-}(t) C(k), \end{aligned} \tag{4.28}$$

where $C(k) = \coth(\omega_{in}(k)/2T_0)/2\omega_{in}(k)$ encodes the initial conditions.

An exact solution can be given[17] in terms of Airy functions. Dimensional analysis shows that, on ignoring the k-dependence of $C(k)$, appropriate for large r (or small k), $\xi_{eq}(\bar{t})$ of Eq.2.5 again sets the scale of the equal-time correlation function. Specifically,

$$G(r, t_0) \propto \int d\kappa\, \frac{\sin \kappa (r/\bar{\xi})}{\kappa (r/\bar{\xi})}\, F(\kappa), \tag{4.29}$$

where $F(0) = 1$ and $F(\kappa) \sim \kappa^{-3}$ for large κ. Kibble's insight is correct, at least for this case of a single (uncoupled) field.

5 Vortex densities do not determine correlation lengths directly

We have seen that there is no reason to disbelieve the causal arguments of Kibble for QFT and Zurek for condensed matter as to the field correlation length at the time of the transition. The excellent agreement with the ^3He experiments also shows that, despite the very interesting simulations of Kopnin et al.[11], this length does, indeed, characterise vortex separation for condensed matter at the time when the defects form.

However, the recent Lancaster experiment shows that this cannot always be the case. Significantly, for ^3He the Ginzburg regime is extremely narrow, whereas for ^4He it is

very broad. In fact, the ^4He experiments begin and end in the Ginzburg regime, where thermal fluctuations dominate. The causality arguments are too simple to accommodate these facts.

If these differences are to be visible in the formalism, it can only be through the way in which we relate vortex density to correlation length. We have already observed that the TDLG equation can be recast as the Fokker-Planck equation, whereby the ensemble averages can be understood as averaging with respect to the probability $p_t[\Phi(\mathbf{x})]$ that, at time t, the field takes value $\Phi(\mathbf{x})$. We can use these probabilities, implicit in the correlation functions, to estimate defect densities.

5.1 Classical defects in condensed matter

It would be foolish to estimate the probability of finding profiles like $\Phi(\mathbf{x})$ of Eq.3.11 directly. One way is to work through *line zeroes*, since vortices have line zeroes of the complex field ϕ at the centre of their cores. The converse is not true since zeroes occur on all scales. However, a starting-point for counting vortices in superfluids is to count line zeroes of an appropriately coarse-grained field, in which structure on a scale smaller than ξ_0, the classical vortex size, is not present[18]. That is, we do not want to entertain vortices within vortices. For the moment, we put in a cutoff $l = O(\xi_0)$ by hand into the Fourier transform $G(k,t)$ of $G(r,t)$, as

$$G_l(r,t) = \int đ^3k\, e^{i\mathbf{k}.\mathbf{r}} G(k,t)\, e^{-k^2l^2}. \tag{5.30}$$

We stress that the *long-distance* correlation length $\xi_{eq}(\bar{t})$ depends essentially on the position of the nearest singularity of $G(k,t)$ in the complex k-plane, *independent* of l.

This is not the case for the line-zero density n_{zero}. For example, in our Gaussian approximation of the previous section n_{zero} can be calculated exactly from the two-point correlation function $G(r,t)$ with $p_t[\Phi]$ implicit. It can be shown, quite easily[19, 20] that it depends on the *short-distance* behaviour of $G_l(r,t)$ as

$$n_{zero}(t) = \frac{-1}{2\pi}\frac{G_l''(0,t)}{G_l(0,t)}, \tag{5.31}$$

the ratio of fourth to second moments of $G(k,t)\, e^{-k^2l^2}$.

There are several prerequisites before line zeroes can be identified with vortex cores, and $n_{zero}(t)$ with $n_{def}(t)$.

- The field, on average must have achieved its symmetry-broken ground-state equilibrium value
$$\langle|\phi|^2\rangle = \alpha_0/\beta. \tag{5.32}$$
This, in itself, is sufficient to show that the causal time $\bar{t}$ is *not* the time to begin looking for defects, since $\langle|\phi|^2\rangle$ is small at this time. This, in turn, requires that $G(k,t)$ be non-perturbatively (in β) large.

- Only when $\partial n_{zero}/\partial l$ is small in comparison to n_{zero}/l at $l = \xi_0$ will the line-zeroes have the non-fractal nature of classical defects on small-scales, although vortices may behave like random walks on larger scales. As the power in the long wavelength modes increases the 'Bragg' peak develops in $k^2 G(k,t)$, moving in towards $k = 0$. This condition then becomes the condition that the peak dominates its tail.

- The energy in field gradients should be commensurate with the energy in classical vortices with same density as that of line zeroes.

We stress that these are necessary, but not sufficient, conditions for classical vortices. In particular, although they can be satisfied in the self-consistent linear approximation that will be outlined below, only the full nonlinearity of the system can establish classical profiles. We will term such zeroes as satisfy these conditions proto-vortices. It has to be said that most (but not all[21, 22]) numerical lattice simulations cannot distinguish between proto-vortices and classical vortices.

5.2 TDLG condensed matter

We begin with condensed matter, which we will find to be easier. As the system evolves away from the transition time, the free equation Eq.4.16 ceases to be valid, to be replaced by the full equation

$$\dot{\phi}_a(\mathbf{x},t) = -[-\nabla^2 + \epsilon(t) + \bar{\beta}|\phi(\mathbf{x},t)|^2]\phi_a(\mathbf{x},t) + \bar{\eta}_a(\mathbf{x},t), \tag{5.33}$$

where $\bar{\beta}$ is the rescaled coupling.

In order to retain some analytic understanding of the way that the density is such an ideal quantity to make predictions for, we adopt the approximation of preserving Gaussian fluctuations by linearising the self-interaction as

$$\dot{\phi}_a(\mathbf{x},t) = -[-\nabla^2 + \epsilon_{eff}(t)]\phi_a(\mathbf{x},t) + \bar{\eta}_a(\mathbf{x},t), \tag{5.34}$$

where ϵ_{eff} contains a (self-consistent) term $O(\bar{\beta}\langle|\phi|^2\rangle)$. Additive renormalisation is necessary, so that $\epsilon_{eff} \approx \epsilon$, as given earlier, for $t \leq t_0$.

Self-consistent linearisation is the standard approximation in non-equilibrium QFT[35, 36], but is not strictly necessary here, since numerical simulations that identify line zeroes of the field can be made that use the full self-interaction[5]. However, there are none that address our particular problems exactly. Given the similarities with the QFT case, for which it is difficult to do much better than a Gaussian, there are virtues in comparing the Gaussian approximation for the two cases.

The solution for $G(r,t)$ is a straightforward generalisation of Eq.4.18,

$$G(r,t) = \int_0^\infty d\tau\, \bar{T}(t-\tau/2)\left(\frac{1}{4\pi\tau}\right)^{3/2} e^{-r^2/4\tau}\, e^{-\int_0^\tau ds\ \epsilon_{eff}(t-s/2)}, \tag{5.35}$$

where $\bar{T}$ is the rescaled temperature, as before.

Assuming a *single* zero of $\epsilon_{eff}(t)$ at $t = t_0$, at $r = 0$ the exponential in the integrand peaks at $\tau = \bar{\tau} = 2(t - t_0)$, the counterpart of the Bragg peak in proper-time. Expanding about $\bar{\tau}$ to quadratic order gives[16]

$$G_l(0,t) \approx \bar{T}_c\, e^{2\int_{t_0}^{t} du\,|\epsilon_{eff}(u)|} \int_0^\infty \frac{d\tau\, e^{-(\tau-2(t-t_0))^2|\epsilon'(t_0)|/4}}{[4\pi(\tau+\bar{l}^2)]^{3/2}}, \tag{5.36}$$

where we have put in the momentum cutoff $k^{-1} > l = \bar{l}\xi_0 = O(\xi_0)$of Eq.5.30 by hand. For times $t > \epsilon_0\tau_Q$ we see that, as the unfreezing occurs, long wavelength modes with $k^2 < t/\tau_Q - \epsilon_0$ grow exponentially.

The effect of the back-reaction is to stop the growth of $G_l(0,t) - G_l(0,t_0) = \langle|\phi|^2\rangle_t - \langle|\phi|^2\rangle_0$ at its symmetry-broken value $\bar{\beta}^{-1}$ in our dimensionless units. A necessary condition for this is $\lim_{u\to\infty}\epsilon_{eff}(u) = 0$. That is, we must choose $\epsilon_{eff}(t) = \epsilon(t) + \bar{\beta}(G_l(0,t) - G_l(0,t_0))$, thereby preserving Goldstone's theorem.

At $t = t_0$, when the approximation Eq.5.36 is good, both numerator and denominator are dominated by the short wavelength fluctuations at small τ. Even though the field is correlated over a distance $\bar{\xi} \gg l$ the density of line zeroes $n_{zero} = O(l^{-2})$ depends entirely on the scale at which we look. In no way would we wish to identify these line zeroes with prototype vortices. However, as time passes the peak of the exponential grows and n_{zero} becomes increasingly insensitive to l. How much time we have depends on the magnitude of $\bar{\beta}$, since once $G(0,t)$ has reached this value it stops growing. The time t^* at which this happens can be estimated by substituting $\epsilon(u)$ for $\epsilon_{eff}(u)$ in the expression for $G_l(0,t)$ above.

For $t > t^*$ the equation for $n_{zero}(t)$ is not so simple since the estimate for $G_l(0,t)$ above, based on a single isolated zero of $\epsilon_{eff}(t)$, breaks down because of the approximate vanishing of $\epsilon_{eff}(t)$ for $t > t^*$. A more careful analysis shows that $G_l(0,t)$ can be written as

$$G_l(0,t) \approx \int_0^\infty \frac{d\tau\, \bar{T}(t-\tau/2)}{[4\pi(\tau+\bar{l}^2)]^{3/2}}\bar{G}(\tau,t), \tag{5.37}$$

where $\bar{G}(\tau,t)$ has the same peak as before at $\tau = 2(t-t_0)$, in magnitude and position, but $\bar{G}(\tau,t) \cong 1$ for $\tau < 2(t-t^*)$. Thus, for $\tau_Q \gg \tau_0$, $G_l(0,t)$ can be approximately separated as $G_l(0,t) \cong G_l^{UV}(t) + G^{IR}(t)$, where

$$G_l^{UV}(t) = \bar{T}(t) \int_0^\infty d\tau\,/[4\pi(\tau+\bar{l}^2)]^{3/2}, \tag{5.38}$$

the counterpart of the Bragg 'tail', describes the scale-*dependent* short wavelength thermal noise, proportional to temperature, and

$$G^{IR}(t) = \frac{\bar{T}_c}{(8\pi(t-t_0))^{3/2}} \int_{-\infty}^{\infty} d\tau \bar{G}(\tau,t) \tag{5.39}$$

describes the scale-*independent*, temperature independent, long wavelength fluctuations. A similar decomposition $G''_l(0,t) \cong G''^{UV}_l(t) + G''^{IR}(t)$ can be performed. In particular, $G''^{IR}(t)/G^{IR}(t) = O(t^{-1})$.

Firstly, suppose that, for $t \geq t^*$, $G^{IR}(t) \gg G_l^{UV}(t)$ and $G''^{IR}(t) \gg G''^{UV}_l(t)$, as would be the case for a temperature quench $\bar{T}(t) \to 0$. Then, with little thermal noise, we have widely separated line zeroes, with density $n_{zero}(t) \approx -G''^{IR}(t)/2\pi G^{IR}(t)$ and $\partial n_{zero}/\partial l$ is small in comparison to n_{zero}/l at $l = \xi_0$. Further, once the line zeroes have straightened on small scales at $t > t^*$, the Gaussian field energy, largely in field gradients, is

$$\bar{F} \approx \langle \int_V d^3x \, \frac{1}{2}(\nabla\phi_a)^2 \rangle = -VG''(0,t), \tag{5.40}$$

where V is the spatial volume. This matches the energy

$$\bar{E} \approx V n_{def}(t)(2\pi G(0,t)) = -VG''(0,t) \tag{5.41}$$

possessed by a network of classical global strings with density n_{zero}, in the same approximation of cutting off their logarithmic tails.

From our comments above, we identify such essentially non-fractal line-zeroes with prototype vortices, and n_{zero} with n_{def}. Of course, we require non-Gaussianity to create true classical energy profiles. Nonetheless, the Halperin-Mazenko result may be well approximated for a while even when the fluctuations are no longer Gaussian[22].

For times $t > t^*$

$$n_{zero}(t) \approx \frac{\bar{t}}{8\pi(t - t_0)} \frac{1}{\xi_0^2} \sqrt{\frac{\tau_0}{\tau_Q}}, \tag{5.42}$$

the solution to Vinen's equation[23]

$$\frac{\partial n_{zero}}{\partial t} = -\chi_2 \frac{\hbar}{m} n_{def}^2, \tag{5.43}$$

where $\chi_2 = 4\pi$ in our approximation[7]. What is remarkable in this approximation is that the density of line zeroes uses *no* property of the self-mass contribution to $\epsilon_{eff}(t)$, self-consistent or otherwise.

This decay law is assumed in the analysis of the Lancaster experiments. The empirical value of χ_2 used in them is not taken from quenches, but turbulent flow experiments. It is suggested[14] that $\chi_2 \approx 0.005$, a good three orders of magnitude smaller than our prediction above. Although the TDLG theory is not very reliable for ^4He, if our estimate is sensible it does imply that vortices produced in a *temperature* quench decay much faster than those produced in turbulence.

Equally importantly, we shall see that, for early time at least, thermal fluctuations are large in the Lancaster experiments. However, for ^3He, with negligable UV contributions, we estimate the primordial density of vortices as

$$n_{zero}(t^*) \approx \frac{\bar{t}}{8\pi(t^* - t_0)} \frac{1}{\xi_0^2} \sqrt{\frac{\tau_0}{\tau_Q}}, \tag{5.44}$$

[7] Calculations for χ_2 for realistic values of ξ_0 and τ_0 give $\chi_2 > 4\pi$ for both ^4He and ^3He

in accord with the original prediction of Zurek. Because of the rapid growth of $G(0,t)$, $(t^* - t_0)/\bar{t} = p > 1 = O(1)$. We note that the factor[8] of $f^2 = 8\pi p$ gives a value of $f = O(10)$, in agreement with the empirical results of [12] and the numerical results of [24][9].

Whereas Eq.5.44 is appropriate for ^3He, the situation for the Lancaster ^4He experiments is complex, since they are *pressure* quenches for which the temperature T is almost *constant* at $T \approx T_c$. Unlike temperature quenches[5, 25], thermal fluctuations here remain at full strength[10]. The necessary time-*independence* of $G^{IR}(t)$ for $t > t^*$ is achieved by taking $\epsilon_{eff}(u) = O(u^{-1})$. In consequence, as t increases beyond t^* the relative magnitude of the UV and IR contributions to $G_l(0,t)$ remains *approximately constant* at its value at $t = t^*$.

Nonetheless, as long as the UV fluctuations are insignificant at $t = t^*$ the density of line zeroes will remain largely independent of scale. This follows if $G''^{IR}(t^*) \gg G''^{UV}_l(t^*)$, since $G''_l(0,t)$ becomes scale-independent later than $G_l(0,t)$. In [15] we showed that this is true provided

$$(\tau_Q/\tau_0)(1 - T_G/T_c) < C\pi^4, \tag{5.45}$$

where $C = O(1)$ and T_G is the Ginzburg temperature. With $\tau_Q/\tau_0 = O(10^3)$ and $(1 - T_G/T_c) = O(10^{-12})$ this inequality is well satisfied for a linearised TDLG theory for ^3He derived[11] from the full TDGL theory[9], but there is no way that it can be satisfied for ^4He, when subjected to a slow mechanical quench, as in the Lancaster experiment, for which $\tau_Q/\tau_0 = O(10^{10})$, since the Ginzburg regime is so large that $(1 - T_G/T_c) = O(1)$. As far as the left hand side of Eq.5.45 is concerned, the ^4He quench is *nineteen* orders of magnitude slower than its ^3He counterpart.

When the inequality is badly violated, as with ^4He for slow pressure quenches, then the density of zeroes $n_{def} = O(l^{-2})$ after t^* again depends explicitly on the scale l at which we look and they are not candidates for vortices. Since the whole of the quench takes place within the Ginzburg regime this is not implausible. However, it is possible that, even though the thermal noise never switches off, there is no more than a postponement of vortex production, since our approximations must break down at some stage. The best outcome is to assume that the effect of the thermal fluctuations on fractal behaviour is diminished, only leading to a delay in the time at which vortices finally appear. Even if we suppose that n_{def} above is a starting point for calculating the density at later times, albeit with a different t_0, thereby preserving Vinen's law, we then have the earlier problem of the large $\chi_2 = O(f^2)$, which would make it almost impossible to see vortices.

For all that, a numerical simulation that goes beyond the Gaussian approximation specifically tailored to the Lancaster parameters is crucial if we are to understand what is really happening. We hope to pursue this elsewhere.

[8] An errant factor of 3 appeared in the result of [15]

[9] The temperature quench of the latter is somewhat different from that considered here, but should still give the same results in this case

[10] Even for ^3He, T/T_c never gets very small, and henceforth we take $T = T_c$ in $G_l(0,t)$ above

[11] Ignoring the position-dependent temperature of [11]

6 The Appearance of Structure in QFT

When, in Section 5.2 we set up the closed time-path formalism for the field probabilities $p_t[\Phi]$, our aim was the limited one of establishing the role of Kibble's causal correlation length $\bar{\xi}$ in Eq.4.29. We now appreciate, from condensed matter theory, that this does not, of itself, imply vortices at that separation.

6.1 Proto-vortices in QFT

To establish a link between the correlation function $G(r,t)$ and vortices is even more problematic in QFT than for condensed matter systems. Yet again, we attempt to count vortices by counting line zeroes[26]. In the Gaussian approximations that we shall continue to adopt the expression Eq.5.31 for n_{zero} is equally applicable to QFT. This counting of zeroes is the basis of numerous numerical simulations[27, 28, 29] of cosmic string networks built from Gaussian fluctuations.

The prerequisites for line zeroes in condensed matter that we posed after Eq.5.31 still stand for QFT (except that $\langle|\phi^2|\rangle = M^2/\lambda$), but there are further complications peculiar to QFT. In particular, in QFT we need to consider the whole density matrix $\langle\Phi'|\rho(t)|\Phi\rangle$ rather than just the diagonal elements $p_t[\Phi] = \langle\Phi|\rho(t)|\Phi\rangle$. Classicality is understood in terms of 'decoherence', manifest most simply by the approximate diagonalisation of the reduced density matrix on coarse-graining. By this we mean the separation of the whole into the 'system', and its 'environment' whose degrees of freedom are integrated over, to give a reduced density matrix. The environment can be either other fields with which our scalar is interacting or even the short wavelength modes of the scalar field itself [30, 31]. When interactions are taken into account this leads to quantum noise and dissipation.

In the Gaussian approximations that we shall adopt here, with $\langle\Phi\rangle = 0$, integrating out short wavelengths with $k > l^{-1}$ is just equivalent to a momentum cut-off at the same value. This gives neither noise nor dissipation and diagonalisation does not occur. Nonetheless, from our viewpoint of counting line-zeroes, fluctuations are still present when $l = O(M^{-1})$ that can prevent us from identifying line-zeroes with proto-vortices, if the quenches are too slow.

For all these caveats, there are other symptoms of classical behaviour once $G_l(0;t)$ is non-perturbatively large. Instead of a field basis, we can work in a particle basis and measure the particle production as the transition proceeds. The presence of a non-perturbatively large peak in $k^2G(k;t)$ at $k = k_0$ signals a non-perturbatively large occupation number $N_{k_0} \propto 1/\lambda$ of particles at the same wavenumber k_0[35]. With n_{zero} of (5.31) of order k_0^2 this shows that the long wavelength modes can now begin to be treated classically. From a slightly different viewpoint, the Wigner functional only peaks about the classical phase-space trajectory once the power is non-perturbatively large[32, 33]. More crudely, the diagonal density matrix elements are only then significantly non-zero for non-perturbatively large field configurations $\phi \propto \lambda^{-1/2}$, like vortices.

6.2 Mode growth v. fluctuations

For early times we revert to the mode decomposition of Eq.4.27. The term $coth(\omega_{in}/2T_0)$ appearing in it can be approximated by $2T_0/\sqrt{\epsilon_0}M$. Even though this is a temperature quench, it shows strong similarities to the pressure quench of condensed matter, since both the long and short wavelength contributions to $G(r,t)$ are scaled by the same temperature and we cannot switch off the latter.

The field becomes ordered, as before, because of the exponential growth of long-wavelength modes, which stop growing once the field has sampled the groundstates. What matters is the relative weight of these modes (the 'Bragg' peak) to the fluctuating short wavelength modes in the decomposition Eq.4.29 at this time, since the contribution of these latter is very sensitive to the cutoff l. Only if their contribution to Eq.3.10 is small when field growth stops can a network of line-zeroes be well-defined at early times, let alone have the predicted density. Since the peak is non-perturbatively large this requires small coupling, which we assume.

Consider a quench with $\epsilon(t)$ as in Eq.4.13, in which the symmetry-breaking begins at relative time $\Delta t = t - t_0 = 0$. For a *free* roll, the exponentially growing modes that appear when $\Delta t > t_k^- = t_Q k^2/M^2$ lead to the approximate WKB solution[34]

$$G(r;\Delta t) \propto \frac{T}{M|m(\Delta t)|}\Big(\frac{M}{\sqrt{\Delta t t_Q}}\Big)^{3/2} e^{\frac{4M\Delta t^{3/2}}{3\sqrt{t_Q}}} e^{-r^2/\xi^2(\Delta t)} \tag{6.46}$$

where $\xi^2(\Delta t) = 2\sqrt{\Delta t t_Q}/M$. The provisional freeze-in time t_* when $\langle|\phi^2|\rangle = M^2/\lambda$ is then, for $Mt_Q < (1/\lambda)$,

$$M\Delta t_* \simeq (Mt_Q)^{1/3}(\ln(1/\lambda))^{2/3} \simeq M\bar{t}(\ln(1/\lambda))^{2/3}, \tag{6.47}$$

where $\Delta t_* = t_* - t_0$. This is greater than $M\bar{t}$, but not by a large multiple. Comparison with condensed matter, for which the ratio is a few $(3-5)$ suggests that we don't need a superweak theory[34].

At this qualitative level the correlation length at t_* is

$$M^2\xi^2(t_*) \simeq (Mt_Q)^{2/3}(\ln(1/\lambda))^{1/3}. \tag{6.48}$$

The effect of the other modes is larger than for the instantaneous quench, giving, at $t = t_*$

$$n_{zero} = \frac{M^2}{\pi(M\tau_Q)^{2/3}}(\ln(1/\lambda))^{-1/3}[1+E]. \tag{6.49}$$

The error term $E = O(\lambda^{1/2}(Mt_Q)^{4/3}(\ln(1/\lambda))^{-1/3})$ is due to oscillatory modes, sensitive to the cutoff. In mimicry of Eq.3.10 it is helpful to rewrite Eq.6.49 as

$$n_{zero} = \Big[\frac{1}{\pi\xi_0^2}\Big(\frac{\tau_0}{\tau_Q}\Big)^{2/3}\Big](\ln(1/\lambda))^{-1/3}[1+E]. \tag{6.50}$$

in terms of the scales $\tau_0 = \xi_0 = M^{-1}$. The first term in Eq.6.50 is the Kibble estimate of Eq.3.10, the second is the small multiplying factor, that yet again shows that estimate can be correct, but for completely different reasons. The third term shows when it can be correct, since E is also a measure of the sensitivity of n_{def} to the scale at which it is measured. The condition $E^2 \ll 1$, necessary for a proto-vortex network to be defined, is then guaranteed if

$$(\tau_Q/\tau_0)^2(1 - T_G/T_c) < C, \tag{6.51}$$

$C = O(1)$, on using the relation $(1 - T_G/T_c) = O(\lambda)$. This is the QFT counterpart to Eq.5.45.

For example, suppose that this approach is relevant to the local strings of a strong Type-II $U(1)$ theory for the early universe, in which the time-temperature relationship $tT^2 = \Gamma M_{pl}$ is valid, where we take $\Gamma = O(10^{-1})$ in the GUT era. If G is Newton's constant and μ the classical string tension then, following [4], $Mt_Q \sim 10^{-1}\lambda^{1/2}(G\mu)^{-1/2}$. The dimensionless quantity $G\mu \sim 10^{-6} - 10^{-7}$ is the small parameter of cosmic string theory. A value $\lambda \sim 10^{-2}$ gives $Mt_Q \sim (Mt_*)^a, a \sim 2$, once factors of π, etc.are taken into account, rather than $Mt_Q \sim 1/\lambda$, and the density of Eq.6.50 may be relevant.

6.3 Backreaction in QFT

To improve upon the free-roll result more honestly, but retain the Gaussian approximation for the field correlation functions, the best we can do is adopt a mean-field approximation along the lines of [35, 36], as we did for the CM systems earlier. As there, it does have the correct behaviour of stopping domain growth as the field spreads to the potential minima. As before, only the large-N expansion preserves Goldstone's theorem.

$G(r;t)$ still has the mode decomposition of Eq.4.29, but the modes $\chi_k^{\pm}$ now satisfy the equation

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t) + \lambda\langle\Phi^2(\mathbf{0})\rangle_t\right]\chi_k^{\pm}(t) = 0, \tag{6.52}$$

where we have taken $N = 2$. Because $\lambda\phi^4$ theory is not asymptotically free, particularly in the Hartree approximation, the renormalised λ coupling shows a Landau ghost. This means that the theory can only be taken as a low energy effective theory.

The end result is[35], on making a single subtraction at $t = 0$, is

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t) + \lambda\int d\!\!{}^-{}^3p\, C(p)[\chi_p^+(t)\chi_p^-(t) - 1]\right]\chi_k^{\pm}(t) = 0. \tag{6.53}$$

which we write as

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 - \mu^2(t)\right]\chi_k(t) = 0. \tag{6.54}$$

On keeping just the unstable modes in $\langle\Phi^2(\mathbf{0})\rangle_t$ then, as it grows, its contribution to (6.53) weakens the instabilities, so that only longer wavelengths become unstable. At t^* the instabilities shut off, by definition, and oscillatory behaviour ensues. Since the mode

with wavenumber $k > 0$ stops growing at time $t_k^+ < t^*$, where $\mu^2(t_k^+) = \mathbf{k}^2$, the free-roll density at t^* must be an overestimate.

An approximation that improves upon the WKB approximation is

$$\chi_k(t) \approx \Big(\frac{\pi M}{2\Omega_k(\eta)}\Big)^{1/2} \exp\Big(\int_0^t dt\,\Omega(t)\Big) \tag{6.55}$$

when $\eta = M(t_k^+ - t) > 0$ is large, and $\Omega_k^2(t) = \mu^2(t) - \mathbf{k}^2$. On expanding the exponent in powers of k and retaining only the quadratic terms we recover the WKB approximation when $\mu(t)$ is non-zero.

The result is that the effect of the back-reaction is to give a time-delay Δt to t^*, corresponding to a decrease in the value $k_0(t)$ at which the power peaks of order

$$\frac{\Delta t}{t^*} = O\Big(\frac{1}{ln(1/\lambda)}\Big). \tag{6.56}$$

The backreaction has little effect for times $t < t^*$. For $t > t^*$ oscillatory modes take over the correlation function and we expect oscillations in $G(k;t)$.

In practice the backreaction rapidly forces $\mu^2(t)$ towards zero if the coupling is not too small[35]. For couplings that are not too weak, this requires that we graft purely oscillatory long wavelength behaviour onto the non-perturbatively large exponential mode

$$\chi_k^+(t^*) \approx \alpha_k \exp\Big(\int_0^{t^*} dt'\mu(t')\Big) \exp\Big(-\frac{\sqrt{\tau_Q t^*}}{M}k^2\Big) \tag{6.57}$$

The end result is a new power spectrum, obtained by superimposing oscillatory behaviour onto the old spectrum, frozen at time t^*. As a gross oversimplification, the contribution from the earlier exponential modes alone can only be to contribute terms something like

$$\begin{aligned} G(r;t) \;&\propto\; \frac{T}{M|m(t^*)|} e^{4M(t^*)^{3/2}/3\sqrt{\tau_Q}} \int_{|\mathbf{k}|<M} d\!\!\!^{-}{}^3k\, e^{i\mathbf{k}.\mathbf{x}}\, e^{-2\sqrt{t^*\tau_Q}k^2/M} \\ &\times\; \Big[\cos k(t-t^*) + \frac{\Omega(k) - W'(k)}{k}\sin k(t-t^*)\Big]^2 \end{aligned} \tag{6.58}$$

to G, where $\Omega = M(t^* - t_k)^{1/2}/\tau_Q^{1/2}$ and $W' = 1/4(t^* - t_k)$. The details are almost irrelevant, since the density of line zeroes is independent of the normalisation, and only weakly dependent on the power spectrum.

The $k = 0$ mode of Eq.6.58 encodes the simple solution $\chi_{k=0}(t) = a + bt$ when $\mu^2 = 0$. As observed[25] by Boyanovsky et al. this has built into it the basic causality discussed by Kibble[3]. Specifically, for $r, t \to \infty$, but $r/2t$ constant ($\neq 1$),

$$G(r,t) \approx \frac{C}{r}\Theta(2t/r - 1). \tag{6.59}$$

It follows directly that this causality, engendered by the Goldstone particles of the self-consistent theory, has little effect on the density of line-zeroes that we expect to mature into fully classical vortices, since that is determined by the behaviour at $r = 0$.

Further, for large t the power spectrum effectively has a k^{-2} behaviour for small k, unlike the white noise that would follow from Eq.6.46. It has been suggested[28] that, for such a spectrum, most, if not all, of the vortices are in loops, with little or no self-avoiding 'infinite' string (but see [29]). If there was no infinite string the evolution of the network could be very different[37] from that of white noise, where approximately 75% of the string is 'infinite'[27]. Although causality due to massless Goldstone modes is unrealistic, the linking of causal behaviour to the long wavelength spectrum is general. It has to be said that this approximation should not be taken very seriously for large t on different grounds, since we would expect rescattering to take place at times $\Delta t = O(1/\lambda)$ in a way that is precluded by the Gaussian approximation.

Returning to our original concerns, if Eq.6.51 is not satisfied, it is difficult to imagine how clean vortices, or proto-vortices, can appear later without some additional ingredient.

7 Conclusions

We examined the Kibble /Zurek predictions for the onset of phase transitions and the appearance of defects (in particular, vortices or global cosmic strings) as a signal of the symmetry breaking. Our results are in agreement with their prediction Eq.3.9 as to the magnitude of the correlation length at the time the transition truly begins, equally true for condensed matter and QFT.

However, this is not simply a measure of the separation of defects at the time of their appearance. The time $\bar{t}$ is too early for the field to have found the true groundstates of the theory. We believe that time, essentially the spinodal time, is the time at which proto-vortices can appear, which can later evolve into the standard classical vortices of the theory.

Even then, they may not appear because of thermal field fluctuations. In TDLG condensed matter thermal noise is proportional to temperature. If temperature is *fixed*, but *not* otherwise, as in the pressure quenches of ^4He, this noise can inhibit the production of vortices, although there are other factors to be taken into account (such as their decay rate). On the other hand, on quenching from a high temperature in QFT there are always thermal fluctuations, and these can also disturb the appearance of vortices. The condition that thermal fluctuations are ignorable at the time that the field has achieved the true groundstates can be written

$$(\tau_Q/\tau_0)^\gamma(1 - T_G/T_c) < C, \tag{7.60}$$

where $\gamma = 1$ for condensed matter and $\gamma = 2$ for QFT. $C = O(1)$.

This restores the role of the Ginzburg temperature T_G that the simple causal arguments overlooked, but does not restore thermal fluctuations as the exclusive agent *for* vortex production, as happened in early arguments. Quenches in ^4He provide the major example for which Eq.7.60 is not satisfied.

What happens at late time is unclear, although for TDLG numerical simulations can be performed (but have yet to address this problem exactly). On the other hand, not only

is the case of a single self-interacting *quantum* scalar field in flat space-time a caricature of the early universe, but it is extremely difficult to go beyond the Gaussian approximation. To do better requires that we do differently. There are several possible approaches. One step is to take the FRW metric of the early universe seriously, whereby the dissipation due to the expansion of the universe can change the situation dramatically[38]. Other approaches are more explicit in their attempts to trigger decoherence explicitly, as we mentioned earlier. Most simply, the short wavelength parts of the field can be treated as an environment to be integrated over, to give a coarse-grained theory of long-wavelength modes acting classically in the presence of noise. However, such noise is more complicated than in TDLG theory, being multiplicative as well as additive, and coloured[39, 31, 30]. This is an area to be pursued elsewhere.

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